Grading guide, Pricing Financial Assets, August 2020 The Exam consists of 4 problems. The first problem will enter the evaluation with a weight of 40%; the others with 20% each.

1. Consider an economy where the value of a bank account B_t at time and the price of a non-dividend paying stock S_t at time t are described by the Ito-processes:

$$dB_t = rB_t dt \;, B_0 = 1$$
$$dS_t = \mu S_t dt + \sigma S_t dz \;, S_0 > 0$$

where μ i constant, σ is a positive constant, r is a constant risk free rate, and dz is the standard short hand notation for a Brownian increment.

a) Why can we use S_t as a numeraire?

b) Consider the process B_t/S_t and express this as an Ito process

c) What μ will make this a martingale?

d) What will the drift rate of lnS_t be under this numeraire?

e) A derivative pays $S_T G(S_T)$ at time $T > t$, where G is some function. With E^S denoting the expectation under the probability measure corresponding to S_t as a numeraire find an expression for the price f_t of this derivative at time t.

Solution:

a) S_t can be used as a numeraire because it is the price of a traded asset with a positive price.

b) The process can be found (directly using $d(\frac{X}{Y}) = \frac{X}{Y}(\frac{dX}{X} - \frac{dY}{Y} - \frac{dX}{X}\frac{dY}{Y} + (\frac{dY}{Y})^2)$ or as here following section 27.3 of Hull (8th ed)) by first considering the process $dlnB_t - dlnS_t$. By Ito's lemma this is

$$(r - \mu + 1/2\sigma^2)dt + \sigma dz$$

Again using Ito's lemma (on the transformation $\exp$) you get

$$d\frac{B_t}{S_t} = (r - \mu + \sigma^2)\frac{B_t}{S_t}dt + \sigma\frac{B_t}{S_t}dz$$

c) This is a martingale if $\mu = r + \sigma^2$.

d) You already found the Ito-process $dlnS_t$. With this value of μ the drift is $r + 1/2\sigma^2$ (whereas it is $r - 1/2\sigma^2$ under the traditional risk neutral measure $\mathbb{Q}$ where B_t is numeraire).

e) The value f_t can be found using the fact that f_t/S_t is a martingale under this measure, thus

$$f_t/S_t = E_t^S[f_T/S_T]$$

and

$$f_t = S_t E_t^S[S_T G(S_T)/S_T] = S_t E_t^S[G(S_T)]$$

2. Let $y(0,T)$ be the continuously compounded yields at time $t = 0$ on zero coupon bonds with maturities $T > 0$, i.e. with prices $P(0,T) = \mathsf{e}^{-y(0,T)T}$.

a) Find the instantaneous forward rate $f(0,T)$ as contracted at 0 for maturity T (i.e. for a period $(T, T+dT)$).

b) Show that if y is increasing in T, then $f(0,T) > y(0,T)$ and interpret this result.

Solution:

a) Using the definition of the instantaneous forward rate you calculate

$$f(0,T) = -\frac{\partial \ln P(0,T)}{\partial T} = -\frac{\partial - y(0,T)T}{\partial T} = \frac{\partial y(0,T)}{\partial T} T + y(0,T)$$

b) The result follows from the assumption that $\frac{\partial y}{\partial T} > 0$. An interpretation is that as long as the forward rate curve is above the yield curve it pulls the yield curve (that averages for the period from 0 to T) up at the margin.

3. Consider a one-period coupon bond that promises to pay its principal (of 1) and the accrued interest from a (continuously compounded) coupon of c at time $t = 1$. Assume that there is a risk free (continuously compounded) interest rate of r.

   Suppose that the bond will default with $\mathbb{Q}$-probability q with a known fractional recovery of $R, 0 < R < 1$ (of principal and accrued interest).

   Use a one-period binomial model to find the coupon c that will give the defaultable bond a par price at time $t = 0$.

   What is the credit spread?

Solution: Since the value of the coupon bond can be found by using the expectation under the risk neutral measure and discounting by the risk free interest rate, c must fulfil:

$$1 = \mathrm{e}^{-r}[qR\mathrm{e}^{c} + (1-q)\mathrm{e}^{c}]$$

with the par value on the left hand side and the present value of the expected cash flow on the right (using risk fee rate for discounting and the risk neutral probability for the expectation).

Solving this for c yields

$$c = r - \ln\left(1 - q(1-R)\right)$$

Thus the credit spread is

$$c - r = -\ln\left(1 - q(1-R)\right) \approx q(1-R)$$

where we for the approximation use that $\ln(1-x) \approx -x$ for small x. Thus the credit spread is approximately the expected loss under the $\mathbb{Q}$-probability.

4. Consider a European call option on a stock with price S_t at time t. Assume that the stock pays no dividends before the expiry of the option.

   a) Define the delta Δ of the call.

b) For an at-the-money call (where the strike K of the option is equal to S_t) the delta is close to $1/2$. Use the Black-Scholes-Merton call option formula to find the strike K where the delta is precisely $1/2$. For your reference the call option formula can be written

$$c_t = S_t \Phi(d_1) - K \mathrm{e}^{-r(T-t)} \Phi(d_2)$$

where

$$d_1 = \frac{\ln \frac{S_t}{K} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

where r is a constant continuously compounded interest rate, T is the expiry date of the option and σ the constant volatility. $\Phi(x)$ denotes the standard normal distribution function in x.

c) Assume $S_t = 1$ and find this strike for parameters $(T - t) = 0.25$, the volatility $\sigma = 0.4$ and the risk free interest rate $r = 0$.

d) What is the delta of the corresponding put option?

Solution:

a) The delta is the partial derivative of the option price with respect to the price of the underlying stock.

b) The delta for a call in the BSM-model is $\Phi(d_1)$. For this to be $1/2$ d_1 must be 0. This is when

$$\ln \frac{S_t}{K} + (r + \tfrac{1}{2}\sigma^2)(T-t) = 0$$

or

$$K = \mathrm{e}^{(r+\frac{1}{2}\sigma^2)(T-t)} S_t$$

c) For the numerical example you get

$$K = \mathrm{e}^{(\frac{1}{2}(0.4)^2)0.25} = \mathrm{e}^{0.02} \approx 1.02$$

i.e. 2 % above the price of the stock.

d) The delta of the put is the delta of the call minus 1 (which can be shown by the put-call-parity).